The Feynman rules

7 kwietnia 2014 08:39

QED

$$\mathcal{L} = \mathcal{L}_0^{Dirac} + \mathcal{L}_0^{e.m.} + \mathcal{L}_1 \quad \text{with} \quad \mathcal{L}_0^{Dirac} = \bar\Psi (i\not\partial - m)\Psi$$

$$\mathcal{L}_0^{e.m.} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$$

$$\bar\Psi [i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\Psi \rightarrow \mathcal{L}_1 = -e\bar\Psi \not{A} \Psi = -e\bar\Psi\gamma^\mu\Psi A_\mu$$

$$\Downarrow$$

$$\alpha = \frac{e^2}{4\pi} \text{ the expansion parameter}$$

QED : $\mathcal{L}_1 = -e :\bar\Psi\gamma^\mu\Psi A_\mu:$

$\mathcal{H}_1 = -\mathcal{L}_1 = e:\bar\Psi\gamma^\mu\Psi A_\mu:$

Heisenberg $\rightarrow$ Interaction : $O'(t) = \underbrace{e^{iH_0^S t} e^{-iHt}}_{U(t)} O^H(t) e^{iHt} e^{-iH_0^S t}$.

$$\mathcal{H}_1' = :e^{iH_0^S t} e^{-iHt} \mathcal{H}_1^H e^{iHt} e^{-iH_0^S t}: = e: U\bar\Psi U^{-1}\gamma^\mu U\Psi U^{-1} U A_\mu U^{-1}: =$$

$$= e : \bar\Psi' \gamma^\mu \Psi' A_\mu' :$$

$$S = 1 + \sum_{n=1}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{(-ie)^n}{n!} \int d^4x_1 \dots d^4x_n \, T\{ :\bar\Psi(x_1)\gamma_{\mu_1}\Psi(x_1)A^{\mu_1}(x_1): \dots :\bar\Psi(x_n)\gamma_{\mu_n}\Psi(x_n)A^{\mu_n}(x_n): \}$$

$$S_{fi} = \langle k_1'\lambda_1', \dots, \bar p_1' \bar s_1', \dots | S | p_1 s_1, \dots, k_1\lambda_1, \dots \rangle$$

$$|p_1 s_1, \dots, k_1\lambda_1, \dots\rangle = a^+_{p_1 s_1} \dots a^+_{k_1\lambda_1} \dots |0\rangle \, (2E_{p_1})^{1/2} \dots (2\omega_{k_1})^{1/2}$$

$$\langle k_1'\lambda_1', \dots, \bar p_1' \bar s_1', \dots | = \langle 0| a_{k_1'\lambda_1'} \dots b_{\bar p_1' \bar s_1'} \dots | \, (2\omega_{k_1})^{1/2} \dots (2E_{\bar p_1})^{1/2}$$

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \sum_{s=1,2} \left(a_{p,s} e^{-ipx} u^s(p) + b^+_{p,s} e^{ipx} v^s(p) \right)$$

$$\bar\Psi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \sum_{s=1,2} \left(b_{p,s} e^{-ipx} \bar v^s(p) + a^+_{p,s} e^{ipx} \bar u^s(p) \right)$$

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3(2\omega_p)^{1/2}} \sum_{\lambda=0}^{3} \left(a_{p,\lambda} e^{-ipx} \epsilon_\mu(p,\lambda) + a^+_{p,\lambda} e^{ipx} \epsilon^*_\mu(p,\lambda) \right)$$

operator matching : electron, positron, photon $\overbrace{\text{require } \Psi^+, \bar\Psi^+, A_\mu^+}^{\text{in the initial state}}$

electron, positron, photon $\underbrace{\text{require } \bar\Psi^-, \Psi^-, A_\mu^-}_{\text{in the final state}}$

$\langle \dots | \quad | a^+_{p,s} |0\rangle$

$\uparrow$

a & Ψ^+

$\langle 0| a_{p,s} | a^+ | \dots \rangle$

$\bar\Psi^-$

$\langle \dots | \, b \, | b^+_{p,s} |0\rangle$

$\bar\Psi^+$

$\langle 0| b_{p,s} | b^+ | \dots \rangle$

Ψ^-

$a \propto \psi^+$ $\qquad\qquad$ $\bar\psi^-$

$\langle \dots | b | b^+_{p,s} | 0 \rangle \qquad \langle 0 | b_{p,s} | b^+ | \dots \rangle$

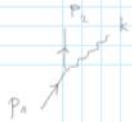
$\bar\psi^+ \qquad\qquad \psi^-$

ψ^+	$\bar\psi^+$	$\bar\psi^-$	ψ^-	A^+_μ	A^-_μ	$S_F(x_2-x_1)$	$D_F(x_2-x_1)$	$-ie\gamma^\mu$
electron absorption	positron absorption	electron emission	positron emission	photon absorption	photon emission			

$\boxed{n=1}$

momentum conservation e.g. $S^{(1)} = \int d^4x \; \bar\psi^- \gamma_\mu \psi^+ A^{\mu -} \propto \int d^4x \; e^{-i(p_1-p_2-k)x} \propto \delta^{(4)}(p_1-p_2-k)$

$\bar\psi^- \to e^{ip_2x}$, $\psi^+ \to e^{-ip_1x}$, $A^{\mu-} \to e^{ikx}$



...

$\Rightarrow \underbrace{\begin{matrix} \pm p_1 \pm p_2 \pm k = 0 \\ p_1^2 = p_2^2 = m^2 \\ k^2 = 0 \end{matrix}}_{\text{can not be satisfied for any sign selection}}$

$\boxed{n=2}$

$$S^{(2)} = \frac{(-ie)^2}{2!} \int d^4x_1 \, d^4x_2 \; T\left\{ :\bar\psi(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1): \; :\bar\psi(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2): \right\}$$

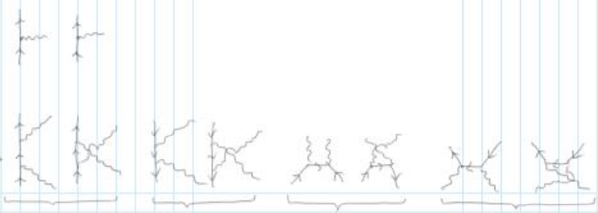
Contractions between operators contained within the scope of a normal product do not contribute to the rhs of the Wick's theorem.

- Show that $\langle 0 | \psi(x_1) \psi(x_2) | 0 \rangle = \langle 0 | \bar\psi(x_1) \bar\psi(x_2) | 0 \rangle = 0$

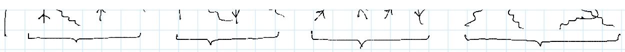
$$S^{(2)} = \frac{(-ie)^2}{2!} \int d^4x_1 \, d^4x_2 \; :\bar\psi(x_1)\gamma_\mu \psi(x_1) \; \bar\psi(x_2)\gamma_\nu \psi(x_2) \; A^\mu(x_1) A^\nu(x_2): \; +$$

(a) — " — $:\bar\psi(x_1)\gamma_\mu \underbrace{\psi(x_1) \; \bar\psi(x_2)}\gamma_\nu \psi(x_2) \; A^\mu(x_1) A^\nu(x_2): \; +$

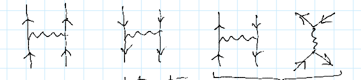
(b) — " — $:\underbrace{\bar\psi(x_1)\gamma_\mu \psi(x_1) \; \bar\psi(x_2)\gamma_\nu \psi(x_2)} \; A^\mu(x_1) A^\nu(x_1):$



(b) — · — : $\psi(x_1)\gamma_\mu\psi(x_1)\ \psi(x_2)\gamma_\nu\psi(x_2)\ A(x_1)A(x_2)$:

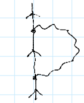


— · — : $\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2)$:

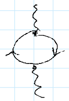


— · — : $\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2)$: +

— · — : $\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2)$: $\Rightarrow$



— · — : $\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2)$:



— · — : $\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2)$:

— Show that (a) and (b) contribute the same

$$\int d^4x_1\, d^4x_2 \Big\{ :\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2): + \qquad (a)$$

$$:\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2): \Big\} \qquad (b)$$

$x_1 \leftrightarrow x_2$ $\to$ $\int d^4x_2\, d^4x_1\ :\bar\psi(x_2)\gamma_\mu\psi(x_2)\ \bar\psi(x_1)\gamma_\nu\psi(x_1)\ A^\mu(x_2)A^\nu(x_1): =$

— Show that $:AB: = \epsilon_{AB} :BA:$ $\qquad A^+|0\rangle = B^+|0\rangle = 0$

$$:(A^+ + A^-)(B^+ + B^-): = A^+B^+ + \epsilon_{AB} B^-A^+ + A^-B^+ + A^-B^- =$$

$$= \epsilon_{AB}\left(\epsilon_{AB}A^+B^+ + B^-A^+ + \epsilon_{AB}A^-B^+ + \epsilon_{AB}A^-B^-\right) =$$

$$= \epsilon_{AB}\left(B^+A^+ + B^-A^+ + :B^+A^-: + B^-A^-\right) = \epsilon_{AB} :BA:$$

$\mu \leftrightarrow \nu$

$$= \int d^4x_1\, d^4x_2\ :\bar\psi(x_1)\gamma_\mu\psi(x_1)\ \bar\psi(x_2)\gamma_\nu\psi(x_2)\ A^\mu(x_1)A^\nu(x_2): \ = \ (b)$$

<u>Compton scattering</u> $\quad e\gamma \to e\gamma$

$|i\rangle = (2E_p)^{1/2}(2\omega_k)^{1/2}\, a^+_{k,\lambda}\, a^+_{p,s}|0\rangle$

$\langle f| = (2E_{p'})^{1/2}(2\omega_{k'})^{1/2}\langle 0|\, a_{k',\lambda'}\, a_{p',s'}$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \sum_{s=1,2} \left(a_{p,s}\, e^{-ipx} u^s(p) + b^+_{p,s}\, e^{ipx} v^s(p) \right)$$

$$\langle f| = (2E_{p'})^{1/2}(2\omega_{k'})^{1/2}\langle 0|\, a_{k',\lambda'}\, a_{p',s'}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \sum_{s=1,2}\left(a_{p,s}\, e^{-ipx} u^s(p) + b^+_{p,s}\, e^{ipx} v^s(p)\right)$$

$$\bar\psi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \sum_{s=1,2}\left(b_{p,s}\, e^{-ipx} \bar v^s(p) + a^+_{p,s}\, e^{ipx} \bar u^s(p)\right)$$

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3(2\omega_p)^{1/2}} \sum_{\lambda=0}^{3}\left(a_{p,\lambda}\, e^{-ipx} \epsilon_\mu(p,\lambda) + a^+_{p,\lambda}\, e^{ipx} \epsilon^*_\mu(p,\lambda)\right)$$

$$S_{fi} = \langle 0|\, a_{k',\lambda'}\, a_{p's'}\, S^{(2)}\, a^+_{k\lambda}\, a^+_{ps}\,|0\rangle \left(2E_p\, 2\omega_p\, 2E_{p'}\, 2\omega_{k'}\right)^{1/2} =$$

$$= 2\,\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\, \langle 0|\, a_{k'\lambda'}\, a_{p's'}\,\big|\, :\bar\psi(x_1)\gamma^\mu \underbrace{\psi(x_1)\ \bar\psi(x_2)}\gamma^\nu \psi(x_2): \times$$

$$\times :\left(A^-_\mu(x_1)\, A^+_\nu(x_2) + A^+_\mu(x_1)\, A^-_\nu(x_2)\right):\, a^+_{k\lambda}\, a^+_{ps}\,|0\rangle\ (\dots)^{1/2} =$$

$$= (\ \ )^{1/2}(-ie)^2 \int d^4x_1\, d^4x_2 \int \frac{d^3p_1}{(2\pi)^3(2E_{p_1})^{1/2}} \int \frac{d^3p_2}{(2\pi)^3(2E_{p_2})^{1/2}} \int \frac{d^3k_1}{(2\pi)^3(2\omega_{k_1})^{1/2}} \int \frac{d^3k_2}{(2\pi)^3(2\omega_{k_2})^{1/2}} \sum_{s_1 s_2=1}^{2}\ \sum_{\lambda_1\lambda_2=0}^{3} \times$$

$$\bar u^{s_1}(p_1)\,\gamma^\mu\, S_F(x_1-x_2)\,\gamma^\nu\, u^{s_2}(p_2)\, e^{ip_1x_1}\, e^{-ip_2x_2}\Big[$$

$$\epsilon^*_\mu(k_1,\lambda_1)\, e^{ik_1x_1}\, \epsilon_\nu(k_2,\lambda_2)\, e^{-ik_2x_2} \langle 0|\, a_{k'\lambda'}\, a_{p's'}\, :a^+_{p_1s_1}\, a_{p_2s_2}\, a^+_{k_1\lambda_1}\, a_{k_2\lambda_2}:\, a^+_{k\lambda}\, a^+_{ps}\,|0\rangle +$$

$$+\ \epsilon_\mu(k_1,\lambda_1)\, e^{-ik_1x_1}\, \epsilon^*_\nu(k_2,\lambda_2)\, e^{ik_2x_2} \langle 0|\, a_{k'\lambda'}\, a_{p's'}\, :a^+_{p_1s_1}\, a_{p_2s_2}\, a_{k_1\lambda_1}\, a^+_{k_2\lambda_2}:\, a^+_{k\lambda}\, a^+_{ps}\,|0\rangle \Big]$$

$$= \langle 0|\, a_{k'\lambda'}\, :a^+_{k_1\lambda_1}\, a_{k_2\lambda_2}:\, a^+_{k\lambda}\,|0\rangle \langle 0|\, a_{p's'}\, :a^+_{p_1s_1}\, a_{p_2s_2}:\, a^+_{ps}\,|0\rangle =$$

$$= (-)(2\pi)^3\,\delta^{(3)}(k_2-k)\,\eta_{\lambda_2\lambda}\,(-)(2\pi)^3\,\delta^{(3)}(k_1-k')\,\eta_{\lambda_1\lambda'}\quad (2\pi)^3\,\delta^{(3)}(p_2-p)\,\delta_{s_2s}\quad (2\pi)^3\,\delta^{(3)}(p_1-p')\,\delta_{s_1s'}$$

$$\to \langle 0|\, a_{k'\lambda'}\, :a_{k_1\lambda_1}\, a^+_{k_2\lambda_2}:\, a^+_{k\lambda}\,|0\rangle\langle 0|\, a_{p's'}\, :a^+_{p_1s_1}\, a_{p_2s_2}:\, a^+_{ps}\,|0\rangle =$$

$$= \langle 0|\, a_{k'\lambda'}\, a^+_{k_2\lambda_2}\, a_{k_1\lambda_1}\, a^+_{k\lambda}\,|0\rangle\langle 0|\, a_{p's'}\, a^+_{p_1s_1}\, a_{p_2s_2}\, a^+_{ps}\,|0\rangle =$$

$$\left\{\begin{array}{l} [a_{k\lambda}, a^+_{k'\lambda'}] = -(2\pi)^3\,\delta^{(3)}(k-k')\,\eta_{\lambda\lambda'} \\ \{a_{ps}, a^+_{p's'}\} = \{b_{ps}, b^+_{p's'}\} = (2\pi)^3\,\delta^{(3)}(p-p')\,\delta_{ss'} \end{array}\right.$$

$$= (-)(2\pi)^3\,\delta^{(3)}(k_1-k)\,\eta_{\lambda_1\lambda}\,(-)(2\pi)^3\,\delta^{(3)}(k_2-k')\,\eta_{\lambda_2\lambda'}\quad (2\pi)^3\,\delta^{(3)}(p_2-p)\,\delta_{s_2s}\quad (2\pi)^3\,\delta^{(3)}(p_1-p')\,\delta_{s_1s'}$$

$$S_{fi} = (\ \ )^{1/2}(-ie)^2 \int d^4x_1\, d^4x_2 \int \frac{d^3p_1}{\cancel{(2\pi)^3}(2E_{p_1})^{1/2}} \int \frac{d^3p_2}{\cancel{(2\pi)^3}(2E_{p_2})^{1/2}} \int \frac{d^3k_1}{\cancel{(2\pi)^3}(2\omega_{k_1})^{1/2}} \int \frac{d^3k_2}{\cancel{(2\pi)^3}(2\omega_{k_2})^{1/2}} \sum_{s_1 s_2=1}^{2}\ \sum_{\lambda_1\lambda_2=0}^{3} \times$$

$$\bar u^{s_1}(p_1)\,\gamma^\mu\, S_F(x_1-x_2)\,\gamma^\nu\, u^{s_2}(p_2)\, e^{ip_1x_1}\, e^{-ip_2x_2}\ \cancel{(2\pi)^{12}}\ \delta^{(3)}(p_1-p')\,\delta^{(3)}(p_2-p)\,\delta_{s_2s}\,\delta_{s_1s'} \times$$

$$\Big[\delta^{(3)}(k_1-k')\,\delta^{(3)}(k_2-k)\,\eta_{\lambda_1\lambda'}\,\eta_{\lambda_2\lambda}\,\epsilon^*_\mu(k_1,\lambda_1)\, e^{ik_1x_1}\,\epsilon_\nu(k_2,\lambda_2)\, e^{-ik_2x_2} + \delta^{(3)}(k_1-k)\,\delta^{(3)}(k_2-k')\,\eta_{\lambda_1\lambda}\,\eta_{\lambda_2\lambda'}\,\epsilon_\mu(k_1,\lambda_1)\, e^{-ik_1x_1}\,\epsilon^*_\nu(k_2,\lambda_2)\, e^{ik_2x_2}\Big] =$$

$$\Big[\delta^{(3)}(k_1-k)\,\delta^{(3)}(k_2-k')\,\eta_{\lambda_1\lambda'}\,\eta_{\lambda_2\lambda}\,\epsilon^*_\mu(k_1,\lambda_1)\,e^{ik_1x_1}\,\epsilon_\nu(k_2,\lambda_2)\,e^{-ik_2x_2} + \delta^{(3)}(k_1-k)\,\delta^{(3)}(k_2-k')\,\eta_{\lambda_1\lambda}\,\eta_{\lambda_2\lambda'}\,\epsilon_\mu(k_1,\lambda_1)\,e^{\cdots}\,\epsilon^*_\nu(k_2,\lambda_2)\,e^{\cdots x_2}\Big] =$$

$$= (-ie)^2\,\cancel{(2E_p\,2\omega_p\,2E_{p'}\,2\omega_{k'})^{1/2}}\int d^4x_1\,d^4x_2 \sum_{\lambda_1\lambda_2} \bar{u}^{s'}(p')\,\gamma^\mu\,S_F(x_1-x_2)\,\gamma^\nu\,u^s(p)\,e^{ip'x_1-ipx_2}\,\times$$

$$\times\Big[\underset{\lambda'}{\epsilon^*_\mu(k',\lambda_1)}\,\underset{\lambda}{\epsilon_\nu(k,\lambda_2)}\,e^{ik'x_1-ikx_2}\,\eta_{\lambda_1\lambda'}\,\eta_{\lambda_2\lambda} + \underset{\lambda}{\epsilon_\mu(k,\lambda_1)}\,\underset{\lambda'}{\epsilon^*_\nu(k',\lambda_2)}\,e^{-ikx_1+ik'x_2}\,\eta_{\lambda_1\lambda}\,\eta_{\lambda_2\lambda'}\Big] =$$

$\lambda,\lambda' = 1,2$ (physical, transverse polarizations

$$= (-ie)^2\int d^4x_1\,d^4x_2\;\bar{u}^{s'}(p')\,\gamma^\mu\,S_F(x_1-x_2)\,\gamma^\nu\,u^s(p)\,e^{ip'x_1-ipx_2}\Big(\epsilon^*_\mu(k',\lambda')\,\epsilon_\nu(k,\lambda)\,e^{ik'x_1-ikx_2} + \epsilon_\mu(k,\lambda)\,\epsilon^*_\nu(k',\lambda')\,e^{-ikx_1+ik'x_2}\Big) =$$

The Feynman rules of QED in the coordinate space

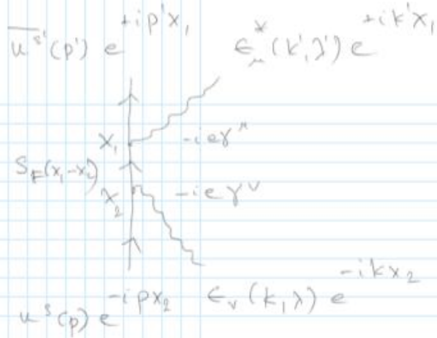
$$S^{(2)}_{fi} = (-ie)^2\int d^4x_1\,d^4x_2\;\bar{u}^{s'}(p')\,\gamma^\mu\,S_F(x_1-x_2)\,\gamma^\nu\,u^s(p)\,e^{ip'x_1-ipx_2}\Big(\epsilon^*_\mu(k',\lambda')\,\epsilon_\nu(k,\lambda)\,e^{ik'x_1-ikx_2} + \epsilon_\mu(k,\lambda)\,\epsilon^*_\nu(k',\lambda')\,e^{-ikx_1+ik'x_2}\Big)$$

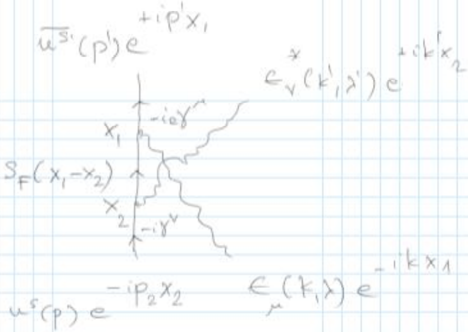
1. To calculate $S^{(n)}_{fi}$ draw all topologically distinct Feynman graphs with n vertices and given initial and final states, for each vertex assign a coordinate variable x_i.
2. Vertex : $-i\gamma_\mu$
3. Internal fermion line: $S_F(x_i-x_j)$
4. Internal photon line : $D_F^{\mu\nu}(x_i-x_j)$
5. External fermion line : - incoming fermion $u^s(p)\,e^{-ipx_i}$
   - incoming antifermion $\bar{v}^s(p)\,e^{+ipx_i}$
   - outgoing fermion $\bar{u}^s(p)\,e^{+ipx_i}$
   - outgoing antifermion $v^s(p)\,e^{-ipx_i}$
6. External photon line : - incoming photon $\epsilon^\mu(k,\lambda)\,e^{-ikx_i}$
   - outgoing photon $\epsilon^{*\mu}(k,\lambda)\,e^{+ikx_i}$

7. Integrate over all x_i

8. Each closed fermion loop receives an extra -1

$$S^{(2)}_{fi} = (-ie)^2 \int d^4x_1\, d^4x_2\, \overline{u}^{s'}(p')\gamma^\mu S_F(x_1-x_2)\gamma^\nu u^s(p)\, e^{ip'x_1 - ipx_2}\left(\epsilon^*_\mu(k',\lambda')\epsilon_\nu(k,\lambda)e^{ik'x_1-ikx_2} + \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k',\lambda')e^{-ikx_1+ik'x_2}\right)$$





The Feynman rules of QED in the momentum space

$$S^{(2)}_{fi} = (-ie)^2 \int d^4x_1\, d^4x_2\, \overline{u}^{s'}(p')\gamma^\mu S_F(x_1-x_2)\gamma^\nu u^s(p)\, e^{ip'x_1 - ipx_2}\left(\epsilon^*_\mu(k',\lambda')\epsilon_\nu(k,\lambda)e^{ik'x_1-ikx_2} + \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k',\lambda')e^{-ikx_1+ik'x_2}\right)$$

$$S_F(x_1-x_2) = \int \frac{d^4\tilde{p}}{(2\pi)^4}\, \frac{i}{\tilde{\not p} - m + i\epsilon}\, e^{-i\tilde{p}(x_1-x_2)}$$

$$= (-ie)^2 \int d^4x_1\, d^4x_2 \int \frac{d^4\tilde{p}}{(2\pi)^4}\, \overline{u}^{s'}(p')\gamma^\mu \frac{i}{\tilde{\not p} - m + i\epsilon}\gamma^\nu u^s(p)\, \times$$

$$\times \left[\epsilon^*_\mu(k',\lambda')\epsilon_\nu(k,\lambda)\, e^{\overbrace{ip'x_1 - ipx_2 + ik'x_1 - ikx_2 - i\tilde{p}(x_1-x_2)}} + \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k',\lambda')\, e^{\overbrace{ip'x_1 - ipx_2 - ikx_1 + ik'x_2 - i\tilde{p}(x_1-x_2)}}\right]$$

$$\hookrightarrow\; ix_1(p'+k'-\tilde{p}) - ix_2(p+k-\tilde{p}) \qquad\qquad \hookrightarrow\; ix_1(p'-k-\tilde{p}) - ix_2(p-k'-\tilde{p})$$

$$\int d^4x_2\, e^{-ix_2(p+k-\tilde{p})} = (2\pi)^4\, \delta^{(4)}(p+k-\tilde{p}) \qquad \int d^4x_2\, e^{-ix_2(p-k'-\tilde{p})} = (2\pi)^4\, \delta^{(4)}(p-k'-\tilde{p})$$

$$\int d^4x_1\, e^{ix_1(p'+k'-\tilde{p})} = (2\pi)^4\, \delta^{(4)}(p'+k'-\tilde{p}) \qquad \int d^4x_1\, e^{ix_1(p'-k-\tilde{p})} = (2\pi)^4\, \delta^{(4)}(p'-k-\tilde{p})$$

$$\int d^4x_2\, e^{\dots} = (2\pi)^4\, \delta^{(4)}(p+k-\tilde{p}) \qquad \int d^4x_2\, e^{\dots} = (2\pi)^4\, \delta^{(4)}(p-k'-\tilde{p})$$

$$\int d^4x_1\, e^{ix_1(p'+k'-\tilde{p})} = (2\pi)^4\, \delta^{(4)}(p'+k'-\tilde{p}) \qquad \int d^4x_1\, e^{ix_1(p'-k-\tilde{p})} = (2\pi)^4\, \delta^{(4)}(p'-k-\tilde{p})$$

$$S_{fi} = (-ie)^2 (2\pi)^4 \int d^4\tilde{p}\; \bar{u}^{s'}(p')\gamma^\mu \frac{i}{\tilde{\not p} - m + i\epsilon}\gamma^\nu u^s(p) \Big[\epsilon^*_\mu(k',\lambda')\,\epsilon_\nu(k,\lambda)\, \delta^{(4)}(p+k-\tilde{p})\,\delta^{(4)}(p'+k'-\tilde{p}) + {}$$
$$+\, \epsilon_\mu(k,\lambda)\,\epsilon^*_\nu(k',\lambda')\,\delta^{(4)}(p-k'-\tilde{p})\,\delta^{(4)}(p'-k-\tilde{p}) \Big] =$$

$$= (-ie)^2 (2\pi)^4 \Big\{ \bar{u}^{s'}(p')\gamma^\mu \frac{i}{(\not k + \not p) - m + i\epsilon}\gamma^\nu u^s(p)\; \epsilon^*_\mu(k',\lambda')\,\epsilon_\nu(k,\lambda) + {}$$
$$+\, \bar{u}^{s'}(p')\gamma^\mu \frac{i}{(\not p - \not k') - m + i\epsilon}\gamma^\nu u^s(p)\; \epsilon_\mu(k,\lambda)\,\epsilon^*_\nu(k',\lambda') \Big\}\, \delta^{(4)}(p+k-p'-k')$$

$$S^{(2)}_{fi} = (-ie)^2 \int \frac{d^4\tilde{p}}{(2\pi)^4}\; \bar{u}^{s'}(p')\gamma^\mu\, S_F(\tilde{p})\,\gamma^\nu u^s(p) \Big\{ \epsilon^*_\mu(k',\lambda')\,\epsilon_\nu(k,\lambda)\,(2\pi)^4\delta^{(4)}(p+k-\tilde{p})\,(2\pi)^4\delta^{(4)}(p'+k'-\tilde{p}) + {}$$
$$+\, \epsilon_\mu(k,\lambda)\,\epsilon^*_\nu(k',\lambda')\,(2\pi)^4\delta^{(4)}(p-k'-\tilde{p})\,(2\pi)^4\delta^{(4)}(p'-k-\tilde{p}) \Big\}$$

1. To calculate $S^{(n)}_{fi}$ draw all topologically distinct Feynman graphs with n vertices and given initial and final states defined by p, s or k, λ, each internal line should have assigned momentum variable p or k.

2. Vertex : $(-ie)\gamma^\mu$

3. Internal fermion line : $S_F(p)$

4. Internal photon line : $D_F^{\mu\nu}(k)$

5. External fermion line : - incoming fermion $u^s(p)$

- outgoing fermion $\bar{u}^s(p)$
- incoming antifermion $\bar{v}^s(p)$
- outgoing antifermion $v^s(p)$

c. External photon line : - incoming photon $\epsilon^\mu(k,\lambda)$

- outgoing photon $\epsilon^{\mu *}(k,\lambda)$

7. Integrate over all internal momenta $\int \frac{d^4p}{(2\pi)^4}$

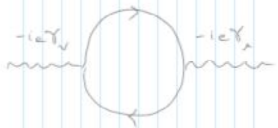
8. Each closed fermion loop receives an extra -1

9. Each vertex enters with 4-momentum conservation $(2\pi)^4 \delta^{(4)}(p'-p\pm k)$

10. Following the above rules one finds $i\mathcal{M}_{fi}$ defined by

$$S_{fi} = \delta_{fi} + \underbrace{(2\pi)^4 \delta^{(4)}(P_i - P_f)\, i\mathcal{M}_{fi}}_{iT_{fi}}$$

-1 for closed fermion loop



$$\longleftrightarrow \frac{(-ie)^2}{2} \int d^4x_1\, d^4x_2 \, : \bar{\psi}(x_1)\gamma_\mu \psi(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2)\, A^\mu(x_1) A^\nu(x_2) :$$

$$\bar{\psi}_\alpha(x_1)(\gamma_\mu)^{\alpha\beta}\psi_\beta(x_1)\, \bar{\psi}_\gamma(x_2)(\gamma_\nu)^{\gamma\delta}\psi_\delta(x_2)$$

$$\psi_\beta(x_1)\bar{\psi}_\gamma(x_2) = S_{F\beta\gamma}(x_1 - x_2)$$

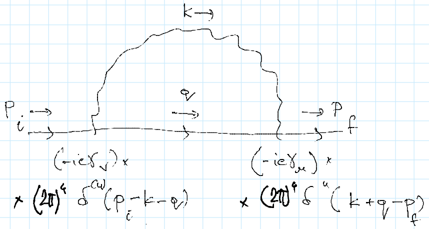
$$\bar{\psi}_\alpha(x_1)\psi_\delta(x_2) = -S_{F\delta\alpha}(x_2 - x_1)$$

$$\bar{\psi}_\alpha(x_1)\psi_\delta(x_2) = \langle 0 | T\{\bar{\psi}_\alpha(x_1)\psi_\delta(x_2)\}|0\rangle = -\langle 0|T\{\psi_\delta(x_2)\bar{\psi}_\alpha(x_1)\}|0\rangle = -S_{F\delta\alpha}(x_2 - x_1)$$

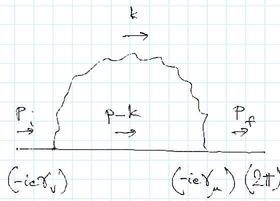
$$S^{(2)} = \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \, (-) S_{F\delta\alpha}(x_2-x_1)(\gamma_\mu)^{\alpha\beta} S_{F\beta\gamma}(x_1-x_2)(\gamma_\nu)^{\gamma\delta} : A^\mu(x_1) A^\nu(x_2) :$$

$$= \frac{(\ldots)}{2!}\int d^4x_1\, d^4x_2\, (-)\, S_{F\delta\alpha}(x_2-x_1)\,(\gamma_\mu)\; S_{F\beta\gamma}(x_1-x_2)\,(\gamma \ldots : A(x_1) A(x_2) :$$

$$= - \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\; \mathrm{Tr}\left\{ S_F(x_2-x_1)\gamma_\mu\, S_F(x_1-x_2)\gamma_\nu \right\} : A^\mu(x_1) A^\nu(x_2)$$

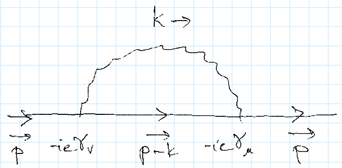


$$\int \frac{d^4k}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4} \longrightarrow \int \frac{d^4k}{(2\pi)^4}$$

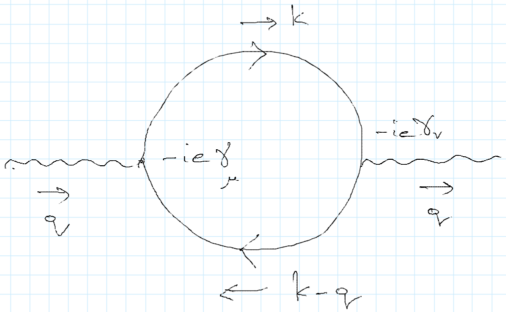


$$(2\pi)^4 \delta^{(4)}(p_i-p_f) \propto (2\pi)^4 \delta^{(4)}(p_i-p_f) \cdot \mathcal{M}_{fi}$$

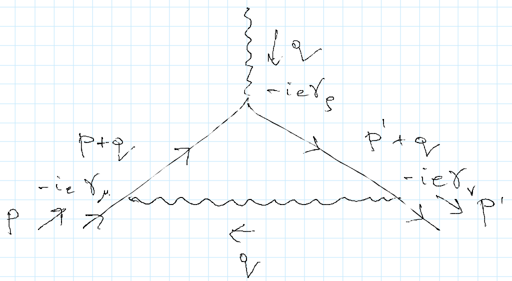
from Feynman rules



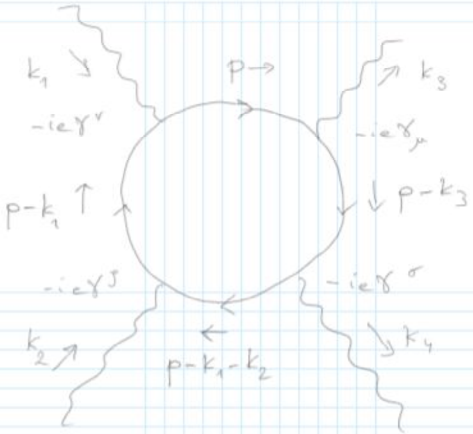
$$i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4}\left(-\frac{i\eta^{\mu\nu}}{k^2+i\varepsilon}\right)(-ie\gamma_\mu)\,\frac{i}{\not p-\not k+m+i\varepsilon}\,(-ie\gamma_\nu)$$



$$i\Pi_{\mu\nu}(q) = (-1)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{(-ie\gamma_\mu)\,\frac{i}{\not k-\not q-m+i\varepsilon}\,(-ie\gamma_\nu)\,\frac{i}{\not k-m+i\varepsilon}\right\}$$

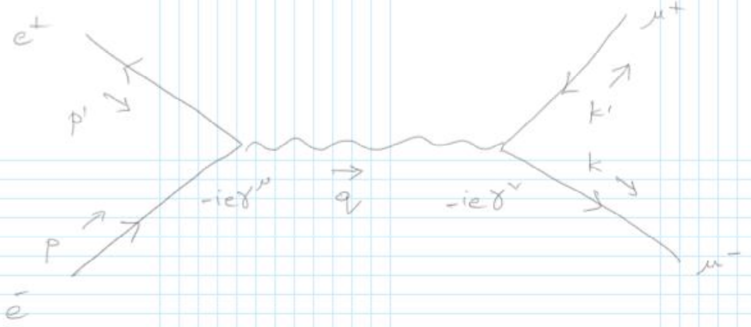


$$-i\Gamma_\mu(p,q) = \int \frac{d^4q}{(2\pi)^4}(-ie\gamma_\nu)\,\frac{i}{\not p'+\not q-m+i\varepsilon}\,(-ie\gamma_\rho)\,\frac{i}{\not p+\not q-m+i\varepsilon}\,(-ie\gamma_\mu)\left(-\frac{i\eta^{\mu\nu}}{q^2+i\varepsilon}\right)$$



$$A_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4) = (-)(-ie)^4\, i^4 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^\mu \frac{1}{\not{p}-m+i\varepsilon} \gamma^\nu \times \frac{1}{\not{p}-\not{k}_1-m+i\varepsilon} \gamma^\rho \frac{1}{\not{p}-\not{k}_1-\not{k}_2-m+i\varepsilon} \gamma^\sigma \frac{1}{\not{p}-\not{k}_3-m+i\varepsilon} \right\}$$

$$\boxed{e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^- \quad \text{in QED}}$$



$$i\mathcal{M}_{fi} = \bar{v}^{s'}(p')\left(-ie\gamma^\mu\right) u^s(p) \frac{-i}{q^2+i\varepsilon}\eta_{\mu\nu}\, \bar{u}^r(k)\left(-ie\gamma^\nu\right) v^{r'}(k') \qquad \text{for } q = p+p' = k+k'$$

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{s^2}\left[\bar{u}(\mu^-)\gamma^\mu v(\mu^+) \cdot \underbrace{v(\mu^+)^\dagger\gamma_0}_{\bar{v}(\mu^+)} \underbrace{\gamma_0\gamma^{\nu\dagger}\gamma_0}_{\gamma^\nu} u(\mu^-)\right] \times \left[\bar{v}(e^+)\gamma_\mu u(e^-) \cdot \bar{u}(e^-)\gamma_\nu v(e^+)\right]$$

- initial beam unpolarized
- final spin undetected

$\Big\} \rightarrow$ average over initial spin, sum over final spin

Let's write spinor indices explicitly:

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{s^2}\left[\bar{u}_a(\mu^-)(\gamma^\mu)_{ab}\, v_b(\mu^+)\cdot \bar{v}_c(\mu^+)(\gamma^\nu)_{cd}\, u_d(\mu^-)\right]\times\left[\bar{v}_{a'}(e^+)(\gamma_\mu)_{a'b'}\, u_{b'}(e^-)\cdot \bar{u}_{c'}(e^-)(\gamma_\nu)_{c'd'}\, v_{d'}(e^+)\right] =$$

$$= \frac{e^4}{s^2}\left[u(\mu^-)\,\bar{u}(\mu^-)\right]_{da}(\gamma^\mu)_{ab}\left[v(\mu^+)\,\bar{v}(\mu^+)\right]_{bc}(\gamma^\nu)_{cd}\left[v(e^+)\,\bar{v}(e^+)\right]_{d'a'}(\gamma_\mu)_{a'b'}\left[u(e^-)\,\bar{u}(e^-)\right]_{b'c'}(\gamma_\nu)_{c'd'}$$

For $p = (E, 0, 0, p)$ one gets

$$u^s(p) = \begin{pmatrix} \left[(E+p)^{1/2}\frac{1}{2}(1-\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1+\sigma^3)\right]\xi^s \\ \left[(E+p)^{1/2}\frac{1}{2}(1+\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1-\sigma^3)\right]\xi^s \end{pmatrix} \quad \text{for } \xi^1 = \begin{pmatrix}1\\0\end{pmatrix},\ \xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$$

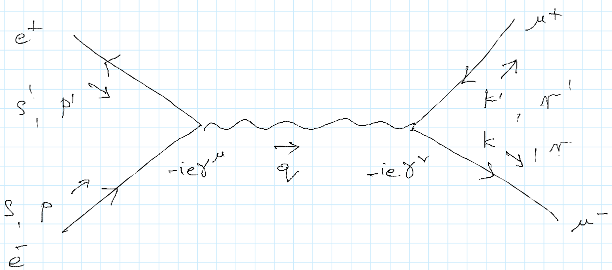
$$v^s(p) = \begin{pmatrix} \left[(E+p)^{1/2}\frac{1}{2}(1-\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1+\sigma^3)\right]\eta^s \\ -\left[(E+p)^{1/2}\frac{1}{2}(1+\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1-\sigma^3)\right]\eta^s \end{pmatrix} \quad \text{for } \eta^1 = \begin{pmatrix}1\\0\end{pmatrix},\ \eta^2 = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\bar{u}^r(p)\, u^s(p) = 2m\,\delta^{rs} \qquad \bar{v}^r(p)\, v^s(p) = -2m\,\delta^{rs}$$

$$u^{r+}(p)\, u^s(p) = 2E_p\,\delta^{rs} \qquad v^{r+}(p)\, v^s(p) = 2E_p\,\delta^{rs}$$

$$\bar{u}^r(p)\, v^s(p) = 0 \qquad \bar{v}^r(p)\, u^s(p) = 0$$

$$\left.\begin{array}{l} \sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \not{p} + m \\ \sum_{s=1,2} v^s(p)\,\bar{v}^s(p) = \not{p} - m \end{array}\right\}\ \text{homework}$$



Dirac equation: $(\not{p} - m)\,u(p) = 0 \qquad \bar{u}(p)(\not{p} - m) = 0$

$(\not{p} + m)\,v(p) = 0 \qquad \bar{v}(p)(\not{p} + m) = 0$

$$(\not p + m)\, v(p) = 0 \qquad \bar v(p)(\not p + m) = 0$$

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{s^2} \left[u(\mu^-)\,\bar u(\mu^-)\right]_{da} (\gamma^\mu)_{ab} \left[v(\mu^+)\,\bar v(\mu^+)\right]_{bc} (\gamma^\nu)_{cd} \left[v(e^+)\,\bar v(e^+)\right]_{d'a'} (\gamma_\mu)_{a'b'} \left[u(e^-)\,\bar u(e^-)\right]_{b'c'} (\gamma_\nu)_{c'd'}$$

$$\frac{1}{2^2} \sum_{\substack{s,s' \\ r,r'}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4s^2} \underbrace{(\not k + m_\mu)_{da} (\gamma^\mu)_{ab} (\not k' - m_\mu)_{bc} (\gamma^\nu)_{cd}}_{\mathrm{Tr}\left\{(\not k + m_\mu)\gamma^\mu(\not k' - m_\mu)\gamma^\nu\right\}} \underbrace{(\not p' - m_e)_{d'a'} (\gamma_\mu)_{a'b'} (\not p + m_e)_{b'c'} (\gamma_\nu)_{c'd'}}_{\mathrm{Tr}\left\{(\not p' - m_e)\gamma_\mu(\not p + m_e)\gamma_\nu\right\}} =$$

$$= \frac{e^4}{4s^2} \mathrm{Tr}\left\{(\not k + m_\mu)\gamma^\mu(\not k' - m_\mu)\gamma^\nu\right\} \mathrm{Tr}\left\{(\not p' - m_e)\gamma_\mu(\not p + m_e)\gamma_\nu\right\}$$

- show that:

a) $\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}$

b) $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$

c) $\mathrm{Tr}(\gamma_{\mu_1}\dots\gamma_{\mu_{2n+1}}) = 0$

d) $\mathrm{Tr}\,\gamma_5 = 0$, $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_5) = 0$

a) $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \quad \rightarrow \quad \mathrm{Tr}\{\gamma_\mu, \gamma_\nu\} = 2\,\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 2\eta_{\mu\nu}\,\mathrm{Tr}\,\mathbb{1}, \quad \mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}$

b) $\mathrm{Tr}(\underbrace{\gamma_\mu\gamma_\nu}_{-\gamma_\nu\gamma_\mu + 2\eta_{\mu\nu}}\gamma_\rho\gamma_\sigma) = \mathrm{Tr}(-\gamma_\nu\underbrace{\gamma_\mu\gamma_\rho}_{-\gamma_\rho\gamma_\mu + 2\eta_{\mu\rho}}\gamma_\sigma) + 2\eta_{\mu\nu}\cdot 4\eta_{\rho\sigma} = \mathrm{Tr}(\gamma_\nu\gamma_\rho\underbrace{\gamma_\mu\gamma_\sigma}_{-\gamma\gamma + 2\eta}) - 2\eta_{\mu\rho}4\cdot\eta_{\nu\sigma} + 2\eta_{\mu\nu}\cdot 4\eta_{\rho\sigma} +$

$-\gamma_\nu\gamma_\mu + 2\eta_{\mu\nu}$ $\qquad -\gamma_\rho\gamma_\mu + 2\eta_{\mu\rho}$ $\qquad -\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}$

$$= -\mathrm{Tr}(\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu) + 2\cdot 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$$

$$\Downarrow$$

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$$

c) $$\mathrm{Tr}(\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}) = \mathrm{Tr}(\gamma_5\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}) = -\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\gamma_5\gamma_{\mu_2}\cdots\gamma_{\mu_{2n+1}}) = (-1)^{2n+1}\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}\gamma_5) =$$

$\{\gamma_5, \gamma_\mu\} = 0$

$$= -\mathrm{Tr}(\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}})$$

d) $$\mathrm{Tr}(\gamma_5) = \mathrm{Tr}(\gamma_0\gamma_0\gamma_5) = -\mathrm{Tr}(\gamma_0\gamma_5\gamma_0) = -\mathrm{Tr}(\gamma_5) = 0$$

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_5) = \mathrm{Tr}(\gamma_0^2\gamma_i\gamma_j\gamma_5) = -\mathrm{Tr}(\gamma_0\gamma_i\gamma_j\gamma_5\gamma_0) = -\mathrm{Tr}(\gamma_i\gamma_j\gamma_5) \quad \text{if } \mu = i,\ \nu = j$$

$\gamma_0^2 = 1, \quad \gamma_i^2 = -1 \quad \Leftarrow \quad \{\gamma_\mu, \gamma_\nu\} = 2\cdot\eta_{\mu\nu}$

$$-\mathrm{Tr}(\gamma_k^2\gamma_0\gamma_j\gamma_5) = -(-)^3\,\mathrm{Tr}(\gamma_k\gamma_0\gamma_j\gamma_5\gamma_k) = -\mathrm{Tr}(\gamma_0\gamma_j\gamma_5) \quad \text{if } \mu = 0,\ \nu = j$$

$k \neq j$

$$\frac{1}{4}\sum_{\substack{s,s'\\ r,r'}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4s^2}\,\mathrm{Tr}\left\{(\not k + m_\mu)\gamma^\mu(\not k' - m_\mu)\gamma^\nu\right\}\mathrm{Tr}\left\{(\not p' - m_e)\gamma_\mu(\not p + m_e)\gamma_\nu\right\} =$$

$$= \frac{e^4}{4s^2}\,\mathrm{Tr}\left(\not k\gamma^\mu \not k'\gamma^\nu - m_\mu^2\gamma^\mu\gamma^\nu\right)\mathrm{Tr}\left(\not p'\gamma_\mu \not p\gamma_\nu - m_e^2\gamma_\mu\gamma_\nu\right) =$$

$$= \frac{e^4}{4s^2}\,4^2\left[k^\mu k'^\nu - \eta^{\mu\nu}k.k' + k^\nu k'^\mu - m_\mu^2\eta^{\mu\nu}\right]\times\left[p'_\mu p_\nu - p'.p\,\eta_{\mu\nu} + p'_\nu p_\mu - m_e^2\eta_{\mu\nu}\right] =$$

$$= 4\frac{e^4}{s^2}\left[(k^\mu k'^\nu + k^\nu k'^\mu) - \eta^{\mu\nu}(kk' + m_\mu^2)\right]\left[(p'_\mu p_\nu + p'_\nu p_\mu) - \eta_{\mu\nu}(p'p + m_e^2)\right] =$$

$$s = (p+p')^2 \qquad s = 2m_e^2 + 2pp' \qquad p.p' = \frac{s}{2} - m_e^2$$

$$s = (k+k')^2 \qquad s = 2m_\mu^2 + 2k.k' \qquad k.k' = \frac{s}{2} - m_\mu^2$$

$$p + p' = k + k'$$

$$p - k = k' - p' \Rightarrow p.k = k'p' = E^2 - \bar p.\bar k = E^2 - |\bar p||\bar k|\cos\theta = \frac{s}{4} - \frac{s}{4}\beta_e\beta_\mu\cos\theta = \frac{s}{4}(1-\beta_e\beta_\mu\cos\theta)$$

$$p - k' = k - p' \Rightarrow p.k' = k.p' = E^2 + \bar p\bar k = E^2 + |\bar p||\bar k|\cos\theta = \frac{s}{4}(1+\beta_e\beta_\mu\cos\theta)$$

CM:

$$p^\mu = (E,\bar p) \qquad p'^\mu = (E,-\bar p) \qquad E^2 - \bar p^2 = m_e^2 \qquad |\bar p| = (E^2 - m_e^2)^{1/2} = \left(\frac{s}{4} - m_e^2\right)^{1/2} = \left(\frac{s}{4}\right)^{1/2}\left(1 - \frac{4m_e^2}{s}\right)^{1/2} \equiv \frac{s^{1/2}}{2}\beta_e$$

$$k^\mu = (E,\bar k) \qquad k'^\mu = (E,-\bar k) \qquad E^2 - \bar k^2 = m_\mu^2 \qquad |\bar k| = \left(\frac{s}{4} - m_\mu^2\right)^{1/2} = \left(\frac{s}{4}\right)^{1/2}\left(1 - \frac{4m_\mu^2}{s}\right)^{1/2} \equiv \frac{s^{1/2}}{2}\beta_\mu$$

$$\theta = \measuredangle(\bar p,\bar k) \qquad (2E)^2 = s$$

$$\frac{1}{4}\sum |\mathcal{M}_{fi}|^2 = 4\frac{e^4}{s^2}\left[k.p'\,k'.p + k.p\,k'.p' - k.k'(p'.p + m_e^2) + \right.$$

$$\frac{1}{4}\sum_{\substack{s,s'\\ r,r'}} |\mathcal{M}_{fi}|^2 = 4\frac{e^4}{s^2}\Big[k\cdot p'\, k'\cdot p + k\cdot p\, k'\cdot p' - k\cdot k'(p'\cdot p + m_e^2) + k'\cdot p'\, k\cdot p + k'\cdot p\, k\cdot p' - k\cdot k'(p'\cdot p + m_e^2) - (k\cdot k' + m_\mu^2)\big(2p\cdot p' - 4(p\cdot p' + m_e^2)\big)\Big] =$$

$$= 4\frac{e^4}{s^2}\Big[2(k\cdot p')^2 + 2(k\cdot p)^2 - k\cdot k'\big(\cancel{2p'p} + \cancel{2m_e^2} + \cancel{2pp'} - \cancel{4pp'} - \cancel{4}^{\,2} m_e^2\big) - m_\mu^2\big(-2pp' - 4m_e^2\big)\Big] =$$

$$= 4\frac{e^4}{s^2}\Big[2\frac{s^2}{4^2}(1+\beta_e\beta_\mu\cos\theta)^2 + 2\frac{s^2}{4^2}(1-\beta_e\beta_\mu\cos\theta)^2 + 2m_e^2\Big(\frac{s}{2} - \cancel{m_\mu^2}\Big) + 2m_\mu^2\Big(\frac{s}{2} - \cancel{m_e^2}\Big) + \cancel{4m_e^2m_\mu^2}\Big] =$$

$$= \cancel{4}\frac{e^4}{\cancel{s^2}}\Big[\cancel{2}\cdot\cancel{2}\frac{\cancel{s^2}}{\cancel{4^2}} + \cancel{2}\cdot\cancel{2}\frac{\cancel{s^2}}{\cancel{4^2}}\beta_e^2\beta_\mu^2\cos^2\theta + 4\frac{\cancel{s}}{s}(m_e^2+m_\mu^2)\Big] = e^4\Big[1 + 4\frac{m_e^2+m_\mu^2}{s} + \beta_e^2\beta_\mu^2\cos^2\theta\Big]$$

$$d\sigma = \frac{1}{64\pi^2 s}|\mathcal{M}_{fi}|^2\frac{|k|}{|p|}\,d\Omega \quad \leftarrow \quad |\mathcal{M}_{fi}|^2 \to \frac{1}{4}\sum_{spin}|\mathcal{M}_{fi}|^2$$

$$d\sigma = \underbrace{\left(\frac{e^2}{4\pi}\right)^2}_{\alpha^2}\frac{1}{4s}\,\frac{\frac{s^{1/2}}{2}\beta_\mu}{\frac{s^{1/2}}{2}\beta_e}\,[\quad]\,d\Omega = \frac{\alpha^2}{4s}\left(\frac{1-\frac{4m_\mu^2}{s}}{1-\frac{4m_e^2}{s}}\right)^{1/2}\left[1 + 4\frac{m_e^2+m_\mu^2}{s} + \Big(1-\frac{4m_\mu^2}{s}\Big)\Big(1-\frac{4m_e^2}{s}\Big)\cos^2\theta\right]d\Omega$$

for $s \gg m_e^2, m_\mu^2$

$$d\sigma = \frac{\alpha^2}{4s}(1+\cos^2\theta)\,d\Omega$$

$$\int_0^{2\pi} d\phi \int^{1} \cos^2\theta\, d\cos\theta = 2\pi\, 2$$

$$\sigma = \int d\sigma = \frac{\alpha^2}{4s}\underbrace{\left(4\pi + 2\pi\,\frac{2}{3}\right)}_{4\pi\,\frac{4}{3}} = \frac{\alpha^2\, 4\pi}{3s}$$

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \cos^2\theta \; d\cos\theta = 2\pi\,\frac{2}{3}$$